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Weighted Adjacency Matrix of a Semigraph and its Spectral Analysis

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Abstract

In this paper we introduce weighted adjacency matrix $W(G)$ of a semigraph G and study the related spectral properties similar to the spectral properties of the adjacency matrix of graphs. We prove Sachs type theorem by interpreting first few coefficients of the characteristic polynomial of $W(G)$ in terms of the semigraph parameters. We extend Harary formula for the determinant of the adjacency matrix of graphs to the weighted adjacency matrix of semigraphs. We find bounds for eigenvalues and energy for certain classes of semigraphs.

AMS Subject classification: 05C50, 05C99

Keywords. Weighted adjacency matrix of a semigraph; $W-$characteristic polynomial; $W-$eigenvalues; $W-$energy.

1. Introduction

Semigraph is a natural generalization of the graph introduced by E. Sampathkumar [13] wherein an edge may contain more than two vertices having ***middle*** vertices apart from the usual ***end*** vertices. Several authors [9 - 14] have extended the concepts and results from graph theory to semigraph theory. There have been attempts [12, 14] to obtain elegant and useful matrix representation in terms of different adjacencies in the semigraph. This has been carried out for the $L-$adjacency matrix $L(G)$ of the semigraph in [12] wherein the corresponding spectral analysis in terms of the intrinsic semigraph properties such as the existence of middle vertices and edges of arbitrary lengths was studied.

In this paper by introducing the concept of 'weighted adjacency', we define the weighted adjacency matrix $W(G)$ of the semigraph and explore the resulting spectral theory by studying the corresponding characteristic polynomial, eigenvalues and the energy. We prove Sachs type theorem giving first few coefficients of this characteristic polynomial in terms of number of vertices, edges and triangles in the semigraph.

We find the determinant of the matrix $W(G)$ by extending Harary formula [8] for the determinant of the adjacency matrix of graphs to the W-adjacency matrix of semigraphs. We derive bounds for eigenvalues (called W-eigenvalues) of the characteristic polynomial of $W(G)$ and we obtain these bounds for the complete semigraphs E_k^c and T_{k-1}^1 and the uniform semigraphs $C_{n,m}$ and $K_{n,m}^c$. As in the case of graphs, we define the W-energy of the semigraph G to be the sum of the absolute values of its W-eigenvalues and derive bounds similar to graph energy bounds obtained in [1, 2, 4].

Definition 1.1: A ***semigraph*** G is a pair (V, X), where V is a non-empty set whose elements are called ***vertices*** of G and X is a set of ordered n-tuples called ***edges*** of G of distinct vertices, for various $n \geq 2$, satisfying the following conditions:

SG1: Any two edges have at most one vertex in common.

SG2: Two edges $(u_1, u_2, \dots, u_m)$ and $(v_1, v_2, \dots, v_n)$ are equal if and only if

(i) $m = n$ and

(ii) either $u_i = v_i$ or $u_i = v_{n-i+1}$ for $1 \leq i \leq n$.

Thus the edge $(u_1, u_2, \dots, u_n)$ is the same as $(u_n, u_{n-1}, \dots, u_1)$.

If $E = (v_1, v_2, \dots, v_n)$ is an edge of a semigraph, we say that v_1 and v_n are the ***end vertices*** of edge E and v_i for $2 \leq i \leq n-1$ are the ***middle vertices*** or ***m-vertices*** of the edge E and also the vertices $v_1, v_2, \dots, v_n$ are said to ***belong to the edge*** E. Two vertices u and v , $u \neq v$, in a semigraph are ***adjacent*** if both of them belong to the same edge.

An edge containing at least one m-vertex is called an ***S-edge***, otherwise it is called an ***ordinary edge***. A semigraph with p vertices and q edges is called a (p, q)-semigraph. A ***partial edge*** of an edge $E = (v_{i_1}, v_{i_2}, \dots, v_{i_n})$ is a $(k - j + 1)$-tuple $E' = (v_{i_j}, v_{i_{j+1}}, \dots, v_{i_k})$ where $1 \leq j < k \leq n$. We say that, the partial edge E' has cardinality $k - j + 1$, which we again denoted by $|E'|$. A ***subedge*** of an edge $E = (v_{i_1}, v_{i_2}, \dots, v_{i_n})$ is a k-tuple $E' = (v_{i_{j_1}}, v_{i_{j_2}}, \dots, v_{i_{j_k}})$ where $1 \leq j_1 < j_2 < \cdots < j_k \leq n$.

The number of vertices belonging to an edge E is called the ***cardinality*** of E and is denoted by $|E|$. A partial edge of cardinality 2 is called a ***unit partial edge***. The ***length of an edge*** E is the number of unit partial edges of the edge E and is denoted by $l(E)$. Thus if $E = (v_1, v_2, \dots, v_k)$ then $l(E) = k - 1$ and $|E| = k$. The ***length of a partial edge*** is defined similarly. If the vertices u, v are adjacent, then $(u, \dots, v)$ is a partial edge whose length is denoted by $l(u, v)$.

Three vertices v_i, v_j and v_k are said to form a ***triangle*** in a semigraph G, if they are pairwise adjacent but do not lie on the same edge. If the vertices v_i, v_j and v_k form a triangle in a semigraph then the partial edges $(v_i, \dots, v_j), (v_j, \dots, v_k)$ and

$(v_k, \ldots, v_i)$ are called the ***sides*** of the triangle.

A semigraph is ***complete*** if any two vertices are adjacent and is ***strongly complete*** if it is complete and every vertex is an end vertex of an edge. The complete semigraph on p vertices consisting of a single edge of cardinality p is denoted by E_p^c. The strongly complete semigraph on p vertices containing one edge of cardinality $p-1$ and all other edges of cardinality 2 is denoted by T_{p-1}^1.

A semigraph G is said to be ***r-uniform*** if the cardinality of each edge in G is r. By introducing m number of middle vertices to each edge of the graph C_n, where C_n is the cycle with n vertices, we get a semigraph which is $(m+2)$-uniform which we denote by $C_{n,m}$. Similarly by introducing m number of middle vertices to each edge of the graph K_n, where K_n is the complete graph with n vertices, we get a semigraph which is $(m+2)$-uniform which we denote by $K_{n,m}^c$. More generally, given a graph G, by introducing m number of middle vertices to each edge of the graph G, we obtain a semigraph which is $(m+2)$-uniform which we denote by G_m^c.

A semigraph can be represented diagrammatically in the plane as follows: The edges are represented by the Jordan curve whose end points are end vertices of the edge. The middle vertices of an edge are denoted by small circles placed on the curve in between the end vertices, in the order specified by the edge E. The end vertices of edges which are not middle vertices of any other edges are represented by thick dots. If an m-vertex v of an edge E is an end vertex of another edge $E^{'}$, we draw a small tangent to the circle (representing v) at the end of the edge $E^{'}$. Often this diagram it self is referred to as the semigraph.

The rest of the paper is organized as follows. In Section 2, W-adjacency matrix of a semigraph is introduced along with its characteristic polynomial. Sachs type theorem is proved and also Harary formula is generalized to semigraphs for the determinant of its weighted adjacency matrix. In Section 3, bounds for the W-eigenvalues and W-energy are obtained for some classes of semigraphs.

2. Weighted Adjacency Matrix of a Semigraph and its Characteristic Polynomial

A semigraph may have edges having several vertices including possible middle vertices apart from two end vertices. The L-adjacency matrix $L(G) = (l_{ij})$ of a semigraph G was defined in [12] to reflect this aspect by defining

$$l_{ij} = \begin{cases} 0 & \text{if } v_i \text{ and } v_j \text{ are not adjacent or } v_i = v_j \\ l(v_i, v_j) & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \end{cases}$$

where two vertices v_i and v_j are adjacent if they belong to the same edge. In this paper, we define 'weighted adjacency' in the semigraph by normalizing across all edges of the semigraph.

2.1 Weighted Adjacency matrix of a Semigraph

Definition 2.1: Given a (p,q)-semigraph G, we define its ***weighted adjacency (or W-adjacency) matrix*** $W(G) = (w_{ij})$, where

$$w_{ij} = \begin{cases} 0 & \text{if } v_i \text{ and } v_j \text{ are not adjacent or } v_i = v_j \\ \dfrac{l(v_i, v_j)}{k} & \text{if } v_i \text{ and } v_j \text{ are adjacent lying on an edge of length } k. \end{cases}$$

Note that $W(G)$ is a symmetric $p \times p$ matrix with entries from $\mathbb{Q}$, the field of rationals. Also, if v_i, v_j are end vertices of an edge E, then $w_{ij} = \frac{l(v_i,v_j)}{l(E)} = 1$. We now compute the row sum of weighted adjacency matrix of the complete semigraph E_k^c.

Proposition 2.2: If $W = (w_{ij})$ is the weighted adjacency matrix of the complete semigraph E_k^c on k vertices, then

$$\sum_{i,j} w_{ij} = \frac{k(k+1)}{3}.$$

Proof: Consider the complete semigraph E_k^c with k vertices.

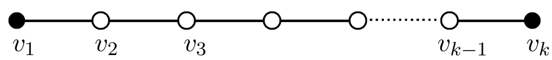

Fig. 1. Semigraph E_k^c

The weighted adjacency matrix of E_k^c is given by

$$W = \frac{1}{k-1} \begin{bmatrix} 0 & 1 & 2 & 3 & \cdots & k-1 \\ 1 & 0 & 1 & 2 & \cdots & k-2 \\ 2 & 1 & 0 & 1 & \cdots & k-3 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ k-1 & k-2 & k-3 & k-4 & \cdots & 0 \end{bmatrix}.$$

Hence

$$\begin{aligned} \sum_{i,j} w_{ij} &= \frac{2}{k-1}\left\{\sum_{1}^{k-1} t + \sum_{1}^{k-2} t + \sum_{1}^{k-3} t + \cdots + \sum_{1}^{1} t\right\} \\ &= \frac{k(k+1)}{3}. \end{aligned}$$ □

Theorem 2.3: If $W = (w_{ij})$ is the weighted adjacency matrix of a semigraph G, then,

$$\sum_{i,j} w_{ij} = \sum_{E\in X} \frac{|E|\ (|E|+1)}{3}.$$

Proof: Each edge E of cardinality k is E_k^c and so by Proposition 2.2 contributes $\frac{k(k+1)}{3}$ to $\sum_{i,j} w_{ij}$ of the W- adjacency matrix of the given semigraph. Thus, summing over all edges E, we obtain,

$$\sum_{i,j} w_{ij} = \sum_{E\in X} \frac{|E|\ (|E|+1)}{3}.$$ □

Corollary 2.4: If $W = (w_{ij})$ is the weighted adjacency matrix of the strongly complete semigraph T^1_{k-1}, then

$$\sum_{i,j} w_{ij} = \frac{(k-1)(k+6)}{3}.$$

Proof: Consider the semigraph T^1_{k-1}.

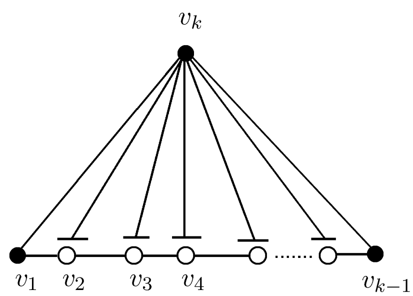

Fig. 2. Semigraph T^1_{k-1}

The semigraph contains one edge of cardinality $k-1$ and the remaining $k-1$ edges of cardinality 2. Hence by Theorem 2.3, we have,

$$\sum_{i,j} w_{ij} = \sum_{E\in X} \frac{|E|\ (|E|+1)}{3} = \frac{k(k-1)}{3} + (k-1)2 = \frac{(k-1)(k+6)}{3}.$$ □

Corollary 2.5: If $W = (w_{ij})$ is the weighted adjacency matrix of the semigraph $C_{n,m}$, then,

$$\sum_{i,j} w_{ij} = \frac{n(m+2)(m+3)}{3}.$$

Proof: Consider the semigraph $C_{n,m}$.

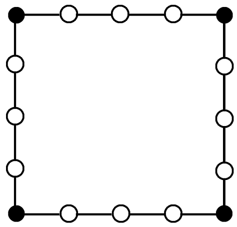

Fig. 3. Semigraph $C_{4,3}$

The semigraph contains n edges, each of cardinality $m+2$. Hence by Theorem 2.3, we have,

$$\sum_{i,j} w_{ij} = \sum_{E\in X} \frac{|E|\ (|E|+1)}{3} = \frac{n(m+2)(m+3)}{3}. \qquad \square$$

Corollary 2.6: If $W = (w_{ij})$ is the weighted adjacency matrix of the semigraph $K^c_{n,m}$, then

$$\sum_{i,j} w_{ij} = \frac{n(n-1)(m+2)(m+3)}{6}.$$

Proof: Consider the semigraph $K^c_{n,m}$.

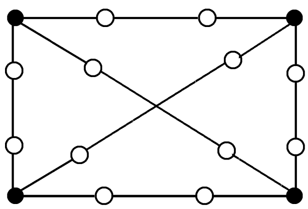

Fig. 4. Semigraph $K^c_{4,2}$

The semigraph contains $\frac{n(n-1)}{2}$ edges, each of cardinality $m+2$. Hence by Theorem 2.3, we have,

$$\sum_{i,j} w_{ij} = \sum_{E\in X} \frac{|E|\ (|E|+1)}{3} = \frac{n(n-1)(m+2)(m+3)}{6}. \qquad \square$$

2.2 The W-Characteristic polynomial

Let W be the weighted adjacency matrix of the given (p,q)-semigraph G. We call the characteristic polynomial of W as the ***W-characteristic polynomial*** of G and it is denoted by $\Theta(G,\eta)$. We write

$$\Theta(G,\eta) = \eta^p + h_1\eta^{p-1} + h_2\eta^{p-2} + h_3\eta^{p-3} + \cdots + h_p.$$

As the entries of the matrix W are rational numbers, the coefficients $h_1, h_2, \dots, h_p$ of the polynomial $\Theta(G,\eta)$ are also rational numbers. The eigenvalues $\eta_1, \eta_2, \dots, \eta_p$ of the matrix W, which are the roots of $\Theta(G,\eta)$, are referred to as the ***W-eigenvalues*** of the semigraph G. Note that $\eta_1, \eta_2, \dots, \eta_p$ are real since the matrix W is real symmetric.

The Coefficients h_1, h_2 and h_3 of $\Theta(G,\eta)$

Now we prove a result similar to Sachs theorem (Theorem 1.3, [6]) by expressing the coefficients h_1, h_2 and h_3 of $\Theta(G,\eta)$ in terms of the semigraph parameters:

Theorem 2.7: For a (p,q)-semigraph $G=(V,X)$, the coefficients h_1, h_2 and h_3 of the characteristic polynomial $\Theta(G,\eta)$ satisfy the following:

(i) $h_1 = 0$,

(ii) $-h_2 = \displaystyle\sum_{E\in X} \frac{(|E|)^2\ (|E|+1)}{12\ (|E|-1)}$,

(iii) $-h_3 = \displaystyle\sum_{T\in\triangle} \frac{2abc}{l(E)\ l(E')\ l(E'')} + \frac{1}{3}\sum_{E\in X}\sum_{i=2}^{l(E)} \frac{i^2(i^2-1)\ (l(E)+1-i)}{(\ l(E))^3}$,

where $\triangle$ denotes the set of all triangles T in the semigraph and a, b, c are the lengths of the sides of the triangle T.

Proof: Since the diagonal elements of the matrix W are zero, trace$(W)=0$. Thus $h_1 = \text{trace}(W) = 0$, proving (i).

Proof of (ii): The coefficient $-h_2$ is the sum of all principal minors of the matrix W with two rows and two columns, having non-zero determinants. Corresponding to the edge E with $|E| = k$, these determinants are of the form:

$$\begin{vmatrix} 0 & \frac{1}{k-1} \\ \frac{1}{k-1} & 0 \end{vmatrix}, \begin{vmatrix} 0 & \frac{2}{k-1} \\ \frac{2}{k-1} & 0 \end{vmatrix}, \dots, \begin{vmatrix} 0 & \frac{k-1}{k-1} \\ \frac{k-1}{k-1} & 0 \end{vmatrix}$$

having the values $-\left(\frac{1}{k-1}\right)^2, -\left(\frac{2}{k-1}\right)^2, \dots, -\left(\frac{k-1}{k-1}\right)^2$ respectively. There are $k-1$ determinants of the first type, $k-2$ of the second type, etc., corresponding to vertices v_i and v_j lying on E such that $l(v_i, v_j) = 1, 2, \dots, k-1$ respectively.

Now taking the sum of above values, we see that the contribution of each edge E of cardinality k towards $-h_2$ is given by

$$\begin{aligned} & (k-1)\left(\frac{1}{k-1}\right)^2 + (k-2)\left(\frac{2}{k-1}\right)^2 + \cdots + 2\left(\frac{k-2}{k-1}\right)^2 + 1\left(\frac{k-1}{k-1}\right)^2 \\ = \ & \frac{k^2(k+1)}{12(k-1)}. \end{aligned}$$

Now summing over all edges E of the semigraph G, we have,

$$-h_2 = \sum_{E\in X} \frac{(|E|)^2\ (|E|+1)}{12\ (|E|-1)}.$$

Proof of (iii): $-h_3$ is the sum of all principal minors of the matrix W with three rows and three columns such that their determinants are non-zero. Such a principal minor corresponds to a set of three vertices v_i, v_j and v_k which are pairwise adjacent. There are two cases to be considered.

If v_i, v_j and v_k do not lie on a single edge, then they form a triangle T with the partial edges $S_1 = (v_i, \ldots, v_j)$, $S_2 = (v_j, \ldots, v_k)$ and $S_3 = (v_k, \ldots, v_i)$. Suppose they are the partial edges of length a, b and c of the edges $E, E^{'}$ and $E^{''}$ respectively.

The corresponding principal minor is of the form

$$\begin{vmatrix} 0 & a/l(E) & b/l(E^{'}) \\ a/l(E) & 0 & c/l(E^{''}) \\ b/l(E^{'}) & c/l(E^{''}) & 0 \end{vmatrix},$$

whose value is $\dfrac{2abc}{l(E)\ l(E^{'})\ l(E^{''})}$. Let $\triangle$ denote the set of all such triangles in the semigraph G. If B_1 denotes the sum of all such principal minors corresponding to all the triangles T in $\triangle$, then $B_1 = \displaystyle\sum_{T\in\triangle} \frac{2abc}{l(E)\ l(E^{'})\ l(E^{''})}$.

Now we consider the case when the vertices v_i, v_j and v_k all lie on a single edge $E = (v_1, v_2, \ldots, v_{m+1})$ of length m with $l(v_i, v_j) = a,\ \ l(v_j, v_k) = b$ and $l(v_i, v_k) = a + b$ such that $a + b \leq m$. The corresponding principal minor is of the form

$$\begin{vmatrix} 0 & a/m & b/m \\ a/m & 0 & (a+b)/m \\ b/m & (a+b)/m & 0 \end{vmatrix},$$

whose value is $\dfrac{2ab(a+b)}{m^3}$. We note that $a, b \in D = \{1, 2, \ldots, m-1\}$ such that $a + b \leq m$. To compute the number of such combinations of v_i, v_j, v_k for the edge $E = (v_1, v_2, \ldots, v_{m+1})$, it is enough to count the number of possibilities of $a + b$ for given a and b satisfying $a, b \in D$ such that $a + b \leq m$. Evidently given a and b satisfying the above, the number of combinations of vertices v_i, v_j, v_k such that $l(v_i, v_j) = a,\ l(v_j, v_k) = b$ and $l(v_i, v_k) = a + b$ is $(m + 1) - (a + b)$. The sum of all such principal minors for valid a and b is $\{(m+1)-(a+b)\}\dfrac{2ab(a+b)}{m^3}$. Hence corresponding to the edge $E = (v_1, v_2, \ldots, v_{m+1})$, the sum of all such principal minors is

$$\sum_{a,b\in D;\, a+b\leq m} \{(m+1)-(a+b)\}\frac{2ab(a+b)}{m^3}.$$

Putting $a+b=i$, the above expression becomes,

$$\begin{aligned}\sum_{a,b\in D;\, a+b\leq m}\{(m+1)-(a+b)\}\frac{2ab(a+b)}{m^3} &= \sum_{i=2}^{m}\sum_{a=1}^{i-1}\{(m+1)-i\}\frac{2ia(i-a)}{m^3}\\ &= \frac{1}{3}\sum_{i=2}^{m}\frac{i^2(i^2-1)(m+1-i)}{m^3}.\end{aligned}$$

Now taking the summation corresponding to all edges and denoting the sum by B_2, we obtain,

$$B_2=\frac{1}{3}\sum_{E\in X}\sum_{i=2}^{l(E)}\frac{i^2(i^2-1)(l(E)+1-i)}{(l(E))^3}.$$

Now $-h_3=B_1+B_2$ and so, we obtain,

$$-h_3=\sum_{T\in\triangle}\frac{2abc}{l(E)\ l(E')\ l(E'')}+\frac{1}{3}\sum_{E\in X}\sum_{i=2}^{l(E)}\frac{i^2(i^2-1)\ (l(E)+1-i)}{(l(E))^3}.$$

This completes the proof. □

Remark 2.8: Since every graph is a semigraph with each edge of cardinality 2, for any graph $G=(V,X)$, we have,

$$-h_2=\sum_{E\in X}\frac{(|E|)^2\ (|E|+1)}{12\ (|E|-1)}=\sum_{E\in X}\frac{(2)^2\ (2+1)}{12\ (2-1)}=\sum_{E\in X}1=q.$$

For a graph G, clearly $B_2=0$. For each triangle T in G, we have $a=b=c=1$, as every edge is of length 1. Thus

$$-h_3=\sum_{T\in\triangle}\frac{2abc}{l(E)\ l(E')\ l(E'')}=\sum_{T\in\triangle}2$$

and thus $-h_3$ is the twice the number of triangles in the graph G. These values of h_2 and h_3 so obtained in terms of the graph parameters are well known elementary facts in the algebraic graph theory (for example, see Proposition 2.3 in Biggs [5]).

Example 2.9:
We compute the coefficients h_2 and h_3 for the semigraph $G=(V,X)$ (see Fig. 5) where $V=\{v_1,v_2,v_3,v_4,v_5,v_6,v_7,v_8\}$ and $X=\{E_1,E_2,\ldots,E_7\}$ with $E_1=\{v_1,v_2,v_3\}$, $E_2=\{v_1,v_8,v_5\}$, $E_3=\{v_1,v_7,v_6\}$, $E_4=\{v_3,v_4\}$, $E_5=\{v_3,v_5\}$, $E_6=\{v_4,v_5\}$ and $E_7=\{v_5,v_6\}$.

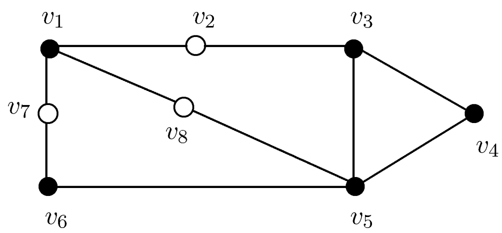

Fig. 5. The Semigraph G

The weighted adjacency matrix of G is given by

$$W(G)=\begin{bmatrix} 0 & 1/2 & 1 & 0 & 1 & 1 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1/2 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1/2 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 \end{bmatrix}.$$

The W-characteristic polynomial of the semigraph G (using *maple*) is:

$$\eta^8-\frac{17}{2}\eta^6-\frac{15}{2}\eta^5+\frac{121}{16}\eta^4+\frac{13}{12}\eta^3-\frac{9}{8}\eta^2-\frac{17}{32}\eta+\frac{1}{16}=0.$$

We now compute the coefficients of η^6 and η^5 by using Theorem 2.7. First we compute the coefficient $-h_2$ of η^6 given by

$$-h_2=\sum_{E\in X}\frac{(|E|)^2\ (|E|+1)}{12\ (|E|-1)}.$$

Note that $|E_1|=|E_2|=|E_3|=3$ and $|E_4|=|E_5|=|E_6|=|E_7|=2$. Summing over three edges of cardinality 3 and four edges of cardinality 2, we get,

$$-h_2=3\left(\frac{9\times 4}{12\times 2}\right)+4\left(\frac{4\times 3}{12\times 1}\right)=\frac{9}{2}+4=\frac{17}{2}.$$

In the semigraph G, there are 3 triangles $T_1=(v_1,v_3,v_5)$, $T_2=(v_1,v_5,v_6)$ and $T_3=(v_3,v_4,v_5)$. For the triangle T_1, the sides are $a_1=2, b_1=1, c_1=2$ lying on the edges E_1,E_5,E_2 respectively; for the triangle T_2, the sides are $a_2=2, b_2=1, c_2=2$ lying on the edges E_2,E_7,E_3 respectively; for the triangle T_3, the sides are $a_3=1, b_3=1, c_3=1$ lying on the edges E_4,E_5,E_6 respectively. Thus,

$$\begin{aligned} B_1 &= \sum_{\triangle\in T}\frac{2abc}{l(E)\ l(E')\ l(E'')} \\ &= \frac{2a_1b_1c_1}{l(E_1)\ l(E_2)\ l(E_5)}+\frac{2a_2b_2c_2}{l(E_2)\ l(E_3)\ l(E_7)}+\frac{2a_3b_3c_3}{l(E_4)\ l(E_5)\ l(E_6)}=2+2+2=6. \end{aligned}$$

We now calculate B_2. Since there are three edges of length 2, we have,

$$B_2 = \frac{1}{3}\sum_{E\in X}\sum_{i=2}^{l(E)} \frac{i^2(i^2-1)\ (l(E)+1-i)}{(l(E))^3} = \frac{1}{3}\times 3\left\{\frac{2^2(2^2-1)\ (1)}{(2)^3}\right\} = \frac{3}{2}.$$

Hence $-h_3 = B_1 + B_2 = 6 + \frac{3}{2} = \frac{15}{2}$.

The Determinant of the W-adjacency matrix of Semigraphs

Given a semigraph G, let G_a denote its associated adjacency graph: G_a has the same vertex set as that of G and two vertices are adjacent in G_a if and only if they are adjacent in the semigraph G. Note that in the semigraph G, two vertices are adjacent if they lie on the same edge. We refer to [12] for a study of the spectral analysis of the adjacency matrix of the adjacency graph G_a.

We give below an example of the semigraph G (given in Fig. 5) and its associated adjacency graph G_a :

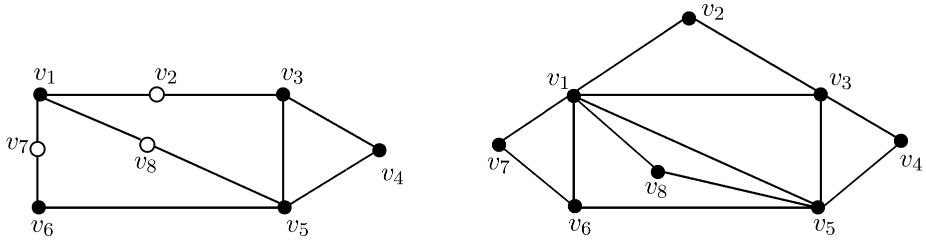

Fig. 6. Semigraph G and its associated adjacency graph G_a

Using the weighted adjacency matrix of the semigraph G, we attach 'weights' for the edges of G_a making G_a into a 'weighted graph'(see p.50 in [3]). If v_i and v_j are adjacent in the semigraph G lying on an edge of length k, then (i, j)-entry w_{ij} in the weighted adjacency matrix $W(G)$ is given by $w_{ij} = l(v_i, v_j)/k$. We assign the weight w_{ij} to the edge v_iv_j in G_a. The resulting weighted graph is denoted by $\overline{G}_a$. For the above semigraph G and the associated adjacency graph G_a, the weighted graph $\overline{G}_a$ is given below:

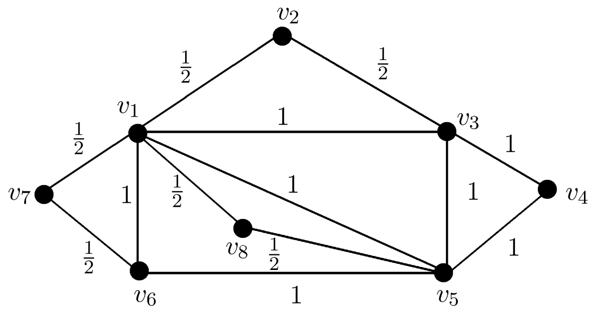

Fig. 7. Weighted graph $\overline{G}_a$

Harary [8] gave an elegant formula for the determinant of the adjacency matrix of a graph G in term of its subgraphs. ***A spanning linear subgraph*** H of a graph G is a subgraph H of G such that $V(H) = V(G)$ and the components of H are single edges or cycles.

Theorem 2.10: Let A be the adjacency matrix of a graph G. Then

$$\det A = \sum_H (-1)^{e(H)} 2^{c(H)},$$

where the summation is over all the spanning linear subgraphs H of G, and $e(H)$ and $c(H)$ denote, respectively, the number of even components and the number of cycles in H. (Here, the even components of H are the components that are either single edges or even cycles of H).

Proof: See Theorem 11.7.2 in [2]. □

We improve upon this theorem to obtain the determinant of the weighted adjacency matrix of a semigraph G:

Theorem 2.11: Let W be the weighted adjacency matrix of the semigraph G. Then, we have,

$$\det W = \sum_H (-1)^{e(H)} 2^{c(H)} \rho(H),$$

where the summation is taken over all spanning linear subgraphs H of the adjacency graph G_a of the semigraph G, $e(H)$ and $c(H)$ denote, respectively, the number of even components and the number of cycles in H, and $\rho(H)$ denotes the product of weights of all the edges of H. (When a single edge is a components of H, we take its contribution to $\rho(H)$ to be $w(e)^2$, where $w(e)$ is the weight of the edge e by considering an edge as a 2-cycle).

Proof: Let the semigraph G be of order n with $V = \{v_1, v_2, \ldots, v_n\}$, and let $W = (w_{ij})$ be its weighted adjacency matrix. let G_a and $\overline{G}_a$ denote respectively its associated adjacency graph and weighted adjacency graph. Note that the weight of the edge v_iv_j in $\overline{G}_a$ is w_{ij}.

A typical term in the expansion of det$W$ is

$$\text{sgn}(\pi)\ w_{1\pi(1)}w_{2\pi(2)}\cdots w_{n\pi(n)},$$

where π is a permutation on $\{1, 2, \ldots, n\}$ and sgn$(\pi) = 1$ or -1 according as π is an even or odd permutation. This term is zero if and only if for some i, $1 \leq i \leq n$, $w_{i\pi(i)} = 0$, i.e., if and only if $\pi(i) = i$ or $\pi(i) = j(\neq i)$ and v_i and v_j are not adjacent in G (or, equivalently v_iv_j is not an edge in the adjacency graph G_a). Thus when this term is nonzero, the permutation π is a product of disjoint cycles of length at least 2. In which case each cycle (ij) of length 2 in π corresponds to the single edge v_iv_j of G_a, while each cycle $(ij\cdots p)$ of length $r > 2$ in π corresponds to a cycle of length r of G_a. Thus each non vanishing term in the expansion of det$W$ gives rise to a spanning linear subgraph H of G_a and conversely. Further in this case the product $w_{1\pi(1)}\cdots w_{n\pi(n)}$ can be rearranged as the product of weights of all edges of H by considering the contribution of a single edge component e of H towards this product as $w(e)^2$, where $w(e)$ is the weight of the edge e. Thus the value of this term is

$$\text{sgn}(\pi) \times \text{product of weights of all edges of } H.$$

Now for any cycle C of S_n, sgn$(C) = 1$ or -1 according as C is an odd or even cycle. Also the sgn(π) is the product of all sgn(C), where C runs through all the cycles in the decomposition of π as the product of disjoint cycles. Thus sgn$(\pi) = (-1)^{e(H)}$, where $e(H)$ is the number of even components of H (i.e., components which are either single edges or even cycles of the subgraph H). Thus the values of this term is

$$(-1)^{e(H)} \times \text{products of weights of all edges of } H.$$

Further, any cycle of H of length $r \geq 3$ gives two orientations and thus each of the undirected cycles of H of length ≥ 3 yield two distinct cycles in S_n. This completes the proof of the theorem. □

Example 2.12:
We find det$W$, the determinant of the W-adjacency matrix of the semigraph G given in Fig. 5 using Theorem 2.11. Note that det$W$ is equal to the constant term of the W-characteristic polynomial of G and hence det$W = 1/16$ (see Example 2.9). The weighted adjacency graph of G is $\overline{G}_a$ given in Fig. 7. Now consider a spanning linear subgraph H_1 of G along with the weights of the edges:

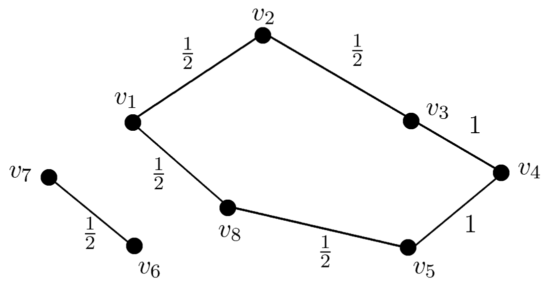

Fig. 8. Spanning linear subgraph H_1

We observe that $e(H_1)$, the number of even components of H_1 is 2; $c(H_1)$, the number of cycles of length ≥ 3 is 1; and $\rho(H_1) = (\frac{1}{2} \times \frac{1}{2}) \cdot (\frac{1}{2} \times \frac{1}{2} \times 1 \times 1 \times \frac{1}{2} \times \frac{1}{2})$. Thus the contribution of H_1 to det$W$ is

$$(-1)^2\ 2^1 \left(\frac{1}{2} \times \frac{1}{2} \times 1 \times 1 \times \frac{1}{2} \times \frac{1}{2}\right) \cdot \left(\frac{1}{2} \times \frac{1}{2}\right) = \frac{1}{32}.$$

Also consider the spanning linear subgraph H_2 of $\overline{G}_a$:

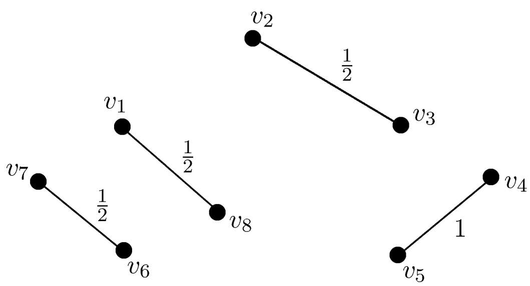

Fig. 9. Spanning linear subgraph H_2

The number of even components of H_2 is 4; the number of cycles of length ≥ 3 is zero; and $\rho(H_2) = (\frac{1}{2} \times \frac{1}{2}) \cdot (\frac{1}{2} \times \frac{1}{2}) \cdot (\frac{1}{2} \times \frac{1}{2}) \cdot (1 \times 1)$. Hence the contribution of H_2 to det$W$ is

$$(-1)^4\ 2^0 \left(\frac{1}{2} \times \frac{1}{2}\right) \cdot \left(\frac{1}{2} \times \frac{1}{2}\right) \cdot \left(\frac{1}{2} \times \frac{1}{2}\right) \cdot (1 \times 1) = \frac{1}{64}.$$

Also consider the spanning linear subgraph H_3 of $\overline{G}_a$:

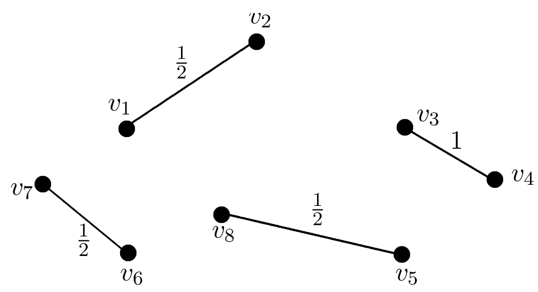

Fig. 10. Spanning linear subgraph H_3

The contribution of H_3 to det$W$ is,

$$(-1)^4\ 2^0 \left(\frac{1}{2}\times\frac{1}{2}\right)\cdot\left(\frac{1}{2}\times\frac{1}{2}\right)\cdot\left(\frac{1}{2}\times\frac{1}{2}\right)\cdot(1\times 1)=\frac{1}{64}.$$

It can be easily seen that H_1, H_2 and H_3 are the only spanning linear subgraphs of $\overline{G}_a$. Thus, we obtain,

$$\det W \ = \ \sum_H (-1)^{e(H)}2^{c(H)}\rho(H) \ = \ \frac{1}{32}+\frac{1}{64}+\frac{1}{64} \ = \ \frac{1}{16}.$$

3. W-eigenvalues and W-energy and their bounds

The W-eigenvalues $\eta_1, \eta_2, \ldots, \eta_p$ of the matrix $W(G)$, which are the roots of $\Theta(G,\eta)$, are real since the matrix W is real symmetric.

Proposition 3.1: If $\eta_1, \ldots, \eta_p$ are the W-eigenvalues of the semigraph G, then,

(i) $\displaystyle\sum_{i=1}^{p}\eta_i = 0;$

(ii) $\displaystyle\sum_{i=1}^{p}\eta_i^2 = -2h_2 = 2\sum_{E\in X}\frac{(|E|)^2\ (|E|+1)}{12\ (|E|-1)}.$

Proof: Since $h_1 = 0$, the sum of the roots of the characteristic equation $\Theta(G,\eta)=0$ is zero, proving (i). Also, we have,

$$\sum_{i=1}^{p}\eta_i^2 = \left(\sum_{i=1}^{p}\eta_i\right)^2 - 2\sum_{i<j}\eta_i\eta_j = -2h_2,$$

and this proves (ii). □

3.1. Bounds for W-eigenvalues of some semigraphs

For any semigraph G, we now give the bounds for the largest W-eigenvalue $\eta_{\max}$. We make use of the well known result in linear algebra: Let A be a real symmetric $n \times n$ matrix with eigenvalues $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$ arranged in nonincreasing order. Let $\|x\|$ denote the usual Euclidean norm: $\|x\| = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$. Then the following extremal representation holds (see p. 7 in [4] and p. 456 in [15]):

$$\lambda_1(A) = \max_{\|x\|=1} \{x^T A x\} \text{ and } \lambda_n(A) = \min_{\|x\|=1} \{x^T A x\}.$$

Theorem 3.2: If G is a (p, q)-semigraph then $\eta_{\max}$ satisfies:

$$\frac{1}{p}\sum_{i,j} w_{ij} = \frac{1}{p}\sum_{E \in X} \frac{|E|(|E|+1)}{3} \leq \eta_{\max} \leq \sqrt{\sum_{E \in X} \frac{|E|^2\ (|E|+1)}{6\ (|E|-1)}} = \sqrt{-2h_2}.$$

Proof: For the real symmetric matrix W, we have, $\eta_{\max} \geq x^T W x$, for any unit vector x. By choosing the unit vector $x = \frac{1}{\sqrt{p}}(1, \ldots, 1)$, we get, $\eta_{\max} \geq (\sum w_{ij})/p$. By Theorem 2.3, we have,

$$\sum_{i,j} w_{ij} = \sum_{E \in X} \frac{|E|\ (|E|+1)}{3}.$$

Thus $\eta_{\max} \geq \frac{1}{3p} \sum_{E \in X} |E|(|E|+1)$.

Using Proposition 3.1, we have,

$$\eta_{\max} \leq \left(\sum_{i=1}^{p} \eta_i^2\right)^{1/2} = \sqrt{-2h_2} = \sqrt{2\sum_{E \in} \frac{(|E|)^2\ (|E|+1)}{12\ (|E|-1)}}.$$

This proves the theorem. □

We now find bounds for $\eta_{\max}$ of the complete semigraphs E_k^c and T_{k-1}^1 by finding the coefficients h_2 and h_3 of their W-characteristic polynomial.

Theorem 3.3: For the semigraph $G = E_k^c$, we have,

$$-h_2 = \frac{k^2(k+1)}{12(k-1)} \quad \text{and} \quad -h_3 = \frac{k^2(k+1)(k^2-4)}{90(k-1)^2}.$$

Further, the maximum eigenvalue $\eta_{\max}$ satisfies:

$$\frac{k+1}{3} \leq \eta_{\max} \leq \sqrt{\frac{k^2(k+1)}{6(k-1)}}.$$

Proof: The semigraph E_k^c has just one edge of cardinality k (see Fig. 1) and so by Theorem 2.7, we have,

$$-h_2 = \sum_{E \in X} \frac{(|E|)^2\ (|E|+1)}{12\ (|E|-1} = \frac{k^2(k+1)}{12(k-1)}.$$

As there are no triangles in E_k^c, $B_1 = 0$. Thus,

$$\begin{aligned} -h_3 = B_2 &= \frac{1}{3}\sum_{i=2}^{l(E)} \frac{i^2(i^2-1)\ (l(E)+1-i)}{(\ l(E))^3} \\ &= \frac{1}{3}\sum_{i=2}^{k-1} \frac{i^2(i^2-1)\ (k-i)}{(k-1)^3} \\ &= \frac{k^2(k+1)(k^2-4)}{90(k-1)^2}. \end{aligned}$$

We now use Theorem 3.2 to find the bounds for $\eta_{\max}$. For the semigraph E_k^c, we have $p = k$ and so,

$$\frac{1}{p}\sum_{E\in X} \frac{|E|(|E|+1)}{3} = \frac{1}{k}\left[\frac{k(k+1)}{3}\right] = \frac{k+1}{3}$$

and

$$\sqrt{\sum_{E\in X} \frac{|E|^2(|E|+1)}{6(|E|-1)}} = \sqrt{\frac{k^2(k+1)}{6(k-1)}}.$$

Now the bounds for $\eta_{\max}$ follows from Theorem 3.2. □

Theorem 3.4: For the semigraph $G = T_{k-1}^1$, we have,

$$-h_2 = (k-1) + \frac{(k-1)^2 k}{12(k-2)} \quad \text{and} \quad -h_3 = \frac{k(k-1)(k^3+27k^2-121k+123)}{90(k-2)^2}.$$

Further, the maximum eigenvalue $\eta_{\max}$ satisfies:

$$\frac{k^2+5k-6}{3k} \leq \eta_{\max} \leq \sqrt{\frac{k^3+10k^2-35k+24}{6(k-2)}}.$$

Proof: Consider the strongly complete semigraph T_{k-1}^1 with k vertices containing one edge of cardinality $k-1$ and the remaining $k-1$ edges of cardinality 2 (see Fig. 2). Using Theorem 2.7 and summing over these k edges, we obtain,

$$-h_2 = \sum_{E\in X} \frac{(|E|)^2\ (|E|+1)}{12\ (|E|-1)} = (k-1) + \frac{(k-1)^2 k}{12(k-2)}.$$

To compute h_3, we first count the number of triangles in T_{k-1}^1. Clearly there are $k-2$ triangles with side lengths $a = b = c = 1$; $k-3$ triangles with side lengths $a = b = 1, c = 2$; and so on and finally one triangle with sides $a = b = 1, c = k-2$. Also, $l(E^{'}) = l(E'') = 1$ and $l(E^{'''}) = k-2$. By Theorem 2.7, we have,

$$\begin{aligned}B_1 &= \sum_{T\in\triangle} \frac{2abc}{l(E')\ l(E'')\ l(E''')}\\ &= \frac{(k-2)2(1)(1)(1)}{k-2} + \frac{(k-3)2(1)(1)(2)}{k-2} + \cdots + \frac{(1)2(1)(1)(k-2)}{k-2}\\ &= \frac{k(k-1)}{3}.\end{aligned}$$

To find B_2, we need only consider the edge of cardinality $k-1$ of the semigraph T^1_{k-1}. This edge is essentially the semigraph E^c_{k-1}. Now we use Theorem 3.3, wherein B_2 for the semigraph E^c_k has been obtained. By replacing k by $k-1$ in Theorem 3.3, we obtain,

$$B_2 = \frac{(k-1)^2\ k\ \{(k-1)^2-4\}}{90(k-2)^2}.$$

Thus,

$$\begin{aligned}-h_3 = B_1 + B_2 &= \frac{k(k-1)}{3} + \frac{(k-1)^2\ k\ \{(k-1)^2-4\}}{90(k-2)^2}\\ &= \frac{k(k-1)(k^3+27k^2-121k+123)}{90(k-2)^2}.\end{aligned}$$

We now use Theorem 3.2 to find the bounds for $\eta_{\max}$. For the semigraph T^1_{k-1}, we have $p = k$ and so,

$$\frac{1}{p}\sum_{E\in X}\frac{|E|(|E|+1)}{3} = \frac{1}{k}\left[(k-1)\frac{2\times 3}{3} + \frac{k(k-1)}{3}\right] = \frac{k^2+5k-6}{3k}$$

and

$$\sqrt{\sum_{E\in X}\frac{|E|^2(|E|+1)}{6(|E|-1)}} = \sqrt{2\left[(k-1) + \frac{k(k-1)^2}{12(k-2)}\right]} = \sqrt{\frac{k^3+10k^2-35k+24}{6(k-2)}}.$$

Now the bounds for $\eta_{\max}$ follows from Theorem 3.2. □

We now find bounds for $\eta_{\max}$ for the semigraphs $C_{n,m}$ and $K^c_{n,m}$ by finding the coefficients h_2 and h_3 of their W-characteristic polynomial.

Theorem 3.5: For the semigraph $G = C_{n,m}$, we have,

$$-h_2 = n\frac{(m+2)^2(m+3)}{12(m+1)}$$

and

$$-h_3 = \begin{cases} 2 + \dfrac{m(m+2)^2(m+3)(m+4)}{30(m+1)^2} & \text{if } n = 3\\ \dfrac{nm(m+2)^2(m+3)(m+4)}{90(m+1)^2} & \text{if } n \geq 4.\end{cases}$$

Further, the maximum eigenvalue $\eta_{\max}$ satisfies:

$$\frac{(m+2)(m+3)}{3(m+1)} \leq \eta_{\max} \leq \sqrt{\frac{n(m+2)^2(m+3)}{6(m+1)}}.$$

Proof: The semigraph $C_{n,m}$ has n edges each of cardinality $m+2$ (see Fig. 3, for $G = C_{4,3}$) and thus using Theorem 2.7 and summing over these n edges, we obtain,

$$-h_2 = \sum_{E \in X} \frac{(|E|)^2\ (|E|+1)}{12\ (|E|-1)} = n\frac{(m+2)^2(m+3)}{12(m+1)}.$$

To determine h_3, we find B_1 and B_2. We first note that there is exactly one triangle with side lengths $a = b = c = m+1$ when $n = 3$ and no triangles in the semigraph when $n \geq 4$. Thus, when for $n = 3$ (i.e., for the semigraph $C_{3,m}$), we have, $B_1 = \frac{2(m+1)^3}{(m+1)^3} = 2$ and for $n \geq 4$, we have, $B_1 = 0$.

For finding B_2, we first note that there are n edges and each edge has cardinality $m+2$ and hence each edge is nothing but E^c_{m+2}. Now by Theorem 3.3, the coefficient $-h_3$ of E^c_{m+2} is given by

$$\frac{m(m+2)^2(m+3)(m+4)}{90(m+1)^2}.$$

Thus, we have,

$$B_2 = \frac{nm(m+2)^2(m+3)(m+4)}{90(m+1)^2}.$$

Hence, when $n = 3$, we have,

$$-h_3 = 2 + \frac{m(m+2)^2(m+3)(m+4)}{30(m+1)^2},$$

and for $n \geq 4$, we have

$$-h_3 = \frac{nm(m+2)^2(m+3)(m+4)}{90(m+1)^2}.$$

We now use Theorem 3.2 to find the bounds for $\eta_{\max}$. For the semigraph $C_{n,m}$, we have $p = nm + n$ and so,

$$\frac{1}{p}\sum_{E \in X} \frac{|E|(|E|+1)}{3} = \frac{1}{n(m+1)}\left[\frac{n(m+2)(m+3)}{3}\right] = \frac{(m+2)(m+3)}{3(m+1)}$$

and

$$\sqrt{\sum_{E \in X} \frac{|E|^2(|E|+1)}{6(|E|-1)}} = \sqrt{\frac{n(m+2)^2(m+3)}{6(m+1)}}.$$

Now the bounds for $\eta_{\max}$ follows from Theorem 3.2. □

Theorem 3.6: For the semigraph $G = K^c_{n,m}$, we have,

$$-h_2 = \frac{n(n-1)(m+2)^2(m+3)}{24(m+1)}$$

and

$$-h_3 = \begin{cases} \dfrac{(n-1)m(m+2)^2(m+3)(m+4)}{90(m+1)^2} & \text{if } n = 2 \\ \dfrac{n(n-1)(n-2)}{3} + \dfrac{n(n-1)m(m+2)^2(m+3)(m+4)}{180(m+1)^2} & \text{if } n \geq 3. \end{cases}$$

Further, the maximum eigenvalue $\eta_{\max}$ satisfies:

$$\frac{(n-1)(m+2)(m+3)}{3\{m(n-1)+2\}} \leq \eta_{\max} \leq \sqrt{\frac{n(n-1)(m+2)^2(m+3)}{12(m+1)}}.$$

Proof: The semigraph $K^c_{n,m}$ has $\frac{n(n-1)}{2}$ edges each of cardinality $m+2$ (see Fig. 4, for $G = K^c_{4,2}$) and thus using Theorem 2.7 and summing over all these edges, we obtain,

$$\begin{aligned} -h_2 &= \sum_{E \in X} \frac{(|E|)^2 \ (|E|+1)}{12 \ (|E|-1)} \\ &= \frac{n(n-1)}{2} \left[\frac{(m+2)^2(m+3)}{12(m+1)} \right] \\ &= \frac{n(n-1)(m+2)^2(m+3)}{24(m+1)}. \end{aligned}$$

To determine h_3, we find B_1 and B_2. We first note that there is no triangle when $n = 2$ and so $B_1 = 0$ in this case. When $n \geq 3$, G has $\binom{n}{3}$ number of triangles each of length $m+1$ with side lengths $a = b = c = m+1$. Thus, in this case we have,

$$B_1 = \frac{n(n-1)(n-2)}{6} \left[\frac{2(m+1)^3}{(m+1)^3} \right] = \frac{n(n-1)(n-2)}{3}.$$

For finding B_2, we first note that there are $\frac{n(n-1)}{2}$ number of edges and each edge has cardinality $m+2$ and hence each edge is nothing but E^c_{m+2}. Now by Theorem 3.3, the coefficient $-h_3$ for E^c_{m+2} is given by

$$\frac{m(m+2)^2(m+3)(m+4)}{90(m+1)^2}.$$

Thus, we have,

$$B_2 = \frac{n(n-1)}{2} \left[\frac{m(m+2)^2(m+3)(m+4)}{90(m+1)^2} \right].$$

Hence, when $n = 2$, we have,

$$-h_3 = \frac{(n-1)m(m+2)^2(m+3)(m+4)}{90(m+1)^2},$$

and for $n \geq 3$, we have

$$-h_3 = \frac{n(n-1)(n-2)}{3} + \frac{n(n-1)m(m+2)^2(m+3)(m+4)}{180(m+1)^2}.$$

We now use Theorem 3.2 to find the bounds for $\eta_{\max}$. For the semigraph $K^c_{n,m}$, we have $p = n + m\frac{n(n-1)}{2}$ and so,

$$\begin{aligned}\frac{1}{p}\sum_{E\in X}\frac{|E|(|E|+1)}{3} &= \frac{1}{(n+m\frac{n(n-1)}{2})}\left[\frac{n(n-1)(m+2)(m+3)}{6}\right] \\ &= \frac{(n-1)(m+2)(m+3)}{3\{m(n-1)+2\}}\end{aligned}$$

and

$$\sqrt{\sum_{E\in X}\frac{|E|^2(|E|+1)}{6(|E|-1)}} = \sqrt{\frac{n(n-1)(m+2)^2(m+3)}{12(m+1)}}.$$

Now the bounds for $\eta_{\max}$ follows from Theorem 3.2. □

3.2. W-Energy bounds for Semigraphs

The energy of a graph is defined as the sum of the absolute values of its eigenvalues. There are several bounds known in literature for the graph energy involving number of its edges and vertices. The McClelland inequality (Theorem 8.7, [6]) gives bounds involving the determinant of the adjacency matrix of the graph. The W-energy of the semigraph G, denoted by $E_W(G)$, is defined to be the sum of the absolute values of the W-eigenvalues of G. Thus $E_W(G) = \sum_{i=1}^{p}|\eta_i|$. We find bounds for the W-energy by extending the McClelland inequality for semigraphs.

Theorem 3.7: For any (p,q)-semigraph G, we have,

$$\sqrt{-2h_2 + p(p-1)|\det W|^{\frac{2}{p}}} \leq E_W(G) \leq \sqrt{-2p\,h_2},$$

where $-h_2$ is the coefficient of η^{p-2} in the W-characteristic polynomial $\Theta(G,\eta)$ of the semigraph G given by: $-h_2 = \sum_{E\in X}\frac{(|E|)^2\ (|E|+1)}{12\ (|E|-1)}$.

Proof: Let $\eta_1, \eta_2, \ldots, \eta_p$ be the W-eigenvalues of the semigraph G. Then,

$$E_W(G)^2 = \left(\sum_{i=1}^{p}|\eta_i|\right)^2 = \sum_{i=1}^{p}\eta_i^2 + \sum_{i\neq j}|\eta_i||\eta_j|.$$

By applying Arithmetic mean - Geometric mean inequality, we obtain,

$$\begin{aligned}\frac{1}{p(p-1)}\sum_{i\neq j}|\eta_i|\,|\eta_j| &\geq \left(\prod_{i\neq j}|\eta_i||\eta_j|\right)^{\frac{1}{p(p-1)}}\\ &= \left(\prod_{i=1}^{p}|\eta_i|^{2(p-1)}\right)^{\frac{1}{p(p-1)}}\\ &= \left|\prod_{i=1}^{p}\eta_i\right|^{\frac{2}{p}}\\ &= |\det W|^{\frac{2}{p}}.\end{aligned}$$

Thus $\sum_{i\neq j}|\eta_i|\,|\eta_j| \geq p(p-1)|\det W|^{\frac{2}{p}}$. Hence from the above, we have,

$$E_W(G)=\left(\sum_{i=1}^{p}\eta_i^2+\sum_{i\neq j}|\eta_i||\eta_j|\right)^{1/2}\geq\sqrt{-2h_2+p(p-1)\,|\det W|^{\frac{2}{p}}}.$$

For determining an upper bound, we make use of the Cauchy-Schwartz's inequality. We have,

$$\begin{aligned}E_W(G)=\sum_{i=1}^{p}|\eta| &\leq \sqrt{p}\left(\sum_{i=1}^{p}|\eta_i|^2\right)^{1/2}\\ &\leq \sqrt{-2p\,h_2}.\end{aligned}$$

This proves the Theorem. □

Now using Theorems 3.3 - 3.7, we prove the following corollary.

Corollary 3.8: The bounds for the W-energy $E_W(G)$ for the semigraphs E_k^c, T_{k-1}^1, $C_{n,m}$ and $K_{n,m}^c$ are given by:

1. The W-energy $E_W(E_k^c)$ of E_k^c lies between

$$\sqrt{\tfrac{k^2(k+1)}{6(k-1)}+k(k-1)|\det W|^{\frac{2}{k}}}\quad\text{and}\quad\sqrt{\tfrac{k^3(k+1)}{6(k-1)}}.$$

2. The W-energy $E_W(T_{k-1}^1)$ of T_{k-1}^1 lies between

$$\sqrt{2(k-1)+\tfrac{(k-1)^2k}{6(k-2)}+k(k-1)|\det W|^{\frac{2}{k}}}\quad\text{and}\quad\sqrt{2k(k-1)+\tfrac{k^2(k-1)^2}{6(k-2)}}.$$

3. The W-energy $E_W(C_{n,m})$ of $C_{n,m}$ lies between

$$\sqrt{\tfrac{n(m+2)^2(m+3)}{6(m+1)}+(p^2-p)\,|\det W|^{\frac{2}{p}}}\quad\text{and}\quad n(m+2)\sqrt{\tfrac{m+3}{6}}$$

where $p = nm + n$, the number of vertices of the semigraph $C_{n,m}$.

4. The W-energy $E_W(K^c_{n,m})$ of $K^c_{n,m}$ lies between

$$\sqrt{\frac{(n^2-n)(m+2)^2(m+3)}{12(m+1)} + (p^2-p)\,|\det W|^{\frac{2}{p}}} \quad \text{and} \quad \sqrt{\frac{2p(n^2-n)(m+2)^2(m+3)}{24(m+1)}}$$

where $p = \frac{n^2m-nm+2n}{2}$, the number of vertices of the semigraph $K^c_{n,m}$. □

Remark 3.9: For the semigraph G given in Fig. 5, the W-eigenvalues (by using *maple*) are (approx.): 3.15022066, 0.89049276, 0.28306960, 0.11014332, −1.69667928, −1.58806609, −0.82943322, −0.31974773. By Theorem 3.2, $2.5 \leq \eta_{\max} \leq 4.12$, whereas $\eta_{\max} = 3.15022066$ (approx.). Also by Theorem 3.7, $6.7 \leq E_W(G) \leq 11.6$, whereas $E_W(G) = 8.86781$ (approx.).

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